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# SENSITIVITY ANALYSIS IN SEQUENCING PROBLEMS

In this paper, three particular sequencing problems are considered. It is shown how to perform the sensitivity analysis for each of these problems. The sensitivity analysis consists in checking how the values of given parameters can vary so that the obtained optimal sequence of jobs remains optimal.

## 1. Introduction

Scheduling has been a topic of great interest since the very beginning of operational research. Over the last fifty years a lot of scheduling models have been developed and a lot of them have found applications in the industry and the scientific research. Sequencing problem is a special case of the more general scheduling problem, in which each schedule can be represented by a sequence of jobs. In a sequencing problem a decision maker is going to find a feasible sequence of jobs for which the value of a given cost function is minimal. A wide review of the sequencing problems together with some complexity results can be found in [2]. In a typical sequencing problem there are given some parameters, the values of which must be fixed before the calculation of an optimal sequence. The set of the parameters almost always includes processing times given for all jobs. Due dates and weights are also typical parameters, often used in sequencing models. After calculation of an optimal sequence it may be important and interesting to check the stability of the obtained solution. In other words, a decision maker may want to check how the values of chosen parameters can vary so that the optimal sequence remains optimal. Such an analysis is called the *sensitivity analysis* and it has already been applied to many mathematical models (for example to linear programming). Some results on the sensitivity analysis in combinatorial optimization, together with a wide literature review can be found in [9].

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In order to find a good method of performing the sensitivity analysis, each particular sequencing problem should be studied separately. In this paper, three well-known sequencing problems are explored. The first two are the single machine sequencing problems denoted in Graham's notation by $1|prec|\max w_iL_i$ and $1|outtree|\Sigma w_iC_i$. The third problem is the two machine permutation flow shop problem denoted by $F2||C_{\max}$. For all of the three considered problems polynomial algorithms have been developed (see [1], [7], [8]). In this paper, some additional results on each of the problems are presented. In particular, for each problem some sufficient and necessary conditions of optimality are recalled. The sufficient and necessary conditions of optimality are then applied to the sensitivity analysis.

## 2. Formulation of the problem

Sequencing problem is a special case of the more general single machine scheduling problem defined, for example, in [2]. Let us start with recalling the general formulation of the sequencing problem. Let $J = \{1, 2, \ldots, n\}$ be a set of jobs ready for processing at time 0. It is assumed that preemption of jobs is not allowed, i.e., the processing of each job cannot be interrupted. The set $J$ may be partially ordered by some precedence constraints $\rightarrow$ between jobs. The set of all the precedence constraints is denoted by *prec* and it can be represented by an acyclic and directed graph without transitive arcs (see [2]). We assume that each solution is represented by a sequence (permutation) of jobs $\pi = (\pi(1), \ldots, \pi(n))$, $\pi(i) \in J$, $i = 1, \ldots, n$. A given sequence $\pi$ is feasible if it preserves all the precedence constraints between jobs, i.e., if $i \rightarrow j$ then job $i$ must appear in $\pi$ before job $j$. We will denote by $\Pi$ the set of all the feasible sequences. Let $\Phi$ be a set of all parameters given in the problem. The set $\Phi$ can include processing times, due dates, weights etc., given for all jobs. There is also given a cost function $F : \Pi \rightarrow \mathbb{R}$. The value of $F(\pi)$ denotes the cost of the sequence $\pi$ for the fixed parameters. For the fixed parameters the sequencing problem consists in calculating a feasible sequence $\pi$ for which the value of the cost function is minimal.

Assume that $\pi$ is a given optimal sequence and $P = \{\alpha_1, \ldots, \alpha_l\} \subseteq \Phi$, $l \geq 1$, is a given subset of parameters. Let us assume that all the parameters in the set $\Phi \backslash P$ are fixed and all the parameters in the set $P$ are variables. Let $V(P; \pi)$ be the set of all vectors $(\alpha_1, \ldots, \alpha_l) \in \mathbb{R}^l$, for which the sequence $\pi$ is optimal. If $|P| = 1$, then we get an important special case in which $V(P; \pi) \subseteq \mathbb{R}$. In this case we obtain the sensitivity range for a chosen single parameter. If $|P| > 1$ then $V(P; \pi)$ should be characterized by some relations between the parameters in the set $P$. In the next three sections we will show how to perform the sensitivity analysis for three well known sequencing problems.

## 3. The problem $1|prec|\max w_iL_i$

In the problem under consideration the set of the parameters includes: nonnegative processing times $p_i$, nonnegative due dates $d_i$ and positive weights $w_i$ given for each job $i \in J$. Thus $\Phi = \{p_1, \ldots, p_n, d_1, \ldots, d_n, w_1, \ldots, w_n\}$. Let us denote by $C_i(\pi)$ the completion time of the $i$th job in a given sequence $\pi$, i.e., if $i = \pi(k)$ then $C_i(\pi) = \sum_{j=1}^{k} p_{\pi(j)}$. The *lateness* $L_i(\pi)$ of job $i$ in the sequence $\pi$ is equal to $C_i(\pi) - d_i$. Finally, the cost function $F(\pi)$ is defined as follows:

$$F(\pi) = \max_{i \in J} w_i L_i(\pi) ,$$

thus the objective is to find a feasible sequence in which the maximal weighted lateness is minimal. The problem can be solved in $O(n^2)$ time by means of Lawler's algorithm [8]. Consider now a feasible sequence $\pi$. A job $k \in J$ is called *critical* in $\pi$ if it has the greatest weighted lateness in $\pi$, i.e.,

$$w_k L_k(\pi) = \max_{i \in J} w_i L_i(\pi) . \tag{1}$$

Note, that there may appear more than one critical job in $\pi$. Let us denote by $\mathbf{S}_i(\pi)$, $i \in J$, the set containing all the jobs processed before job $i$ in $\pi$, which can be moved just after $i$ without violating the precedence constraints. For example, consider the set of jobs $J = \{1, \ldots, 6\}$ with precedence constraints $prec = \{2 \to 4, 2 \to 1, 1 \to 3, 3 \to 6\}$. Let $\pi = (2, 4, 1, 3, \mathbf{5}, 6)$ be a feasible sequence. We obtain $\mathbf{S}_5(\pi) = \{3, 4\}$ since only jobs 3 and 4 can be moved just after job 5 in $\pi$ without violating the precedence constraints. Job 2 cannot be moved just after job 5 because the sequence (4, 1, 3, **5**, **2**, 6) is infeasible and similarly job 1 cannot be moved just after job 5 since the sequence (2, 4, 3, **5**, **1**, 6) is infeasible. The following theorem gives a sufficient and necessary condition of optimality of a given sequence in the considered problem [4]:

**Theorem 1.** *Let $\pi$ be a feasible sequence. Then $\pi$ is optimal if and only if there exists in $\pi$ a critical job $k \in J$ for which the following condition holds:*

$$\forall_{j \in \mathbf{S}_k(\pi)} w_k (C_k(\pi) - d_k) \le w_j (C_k(\pi) - d_j) . \tag{2}$$

Let $k \in J$ be a job in a given feasible sequence $\pi$. Consider the following system $\Upsilon_k(\pi)$:

$$\Upsilon_k(\pi) = \begin{cases} C_{\pi(j)} = p_{\pi(1)} + ... + p_{\pi(j)} & \text{for all } j \in J, \\ w_k(C_k - d_k) \geq w_j(C_j - d_j) & \text{for all } j \in J, \\ w_k(C_k - d_k) \leq w_j(C_k - d_j) & \text{for all } j \in \mathbf{S}_k(\pi). \end{cases} \tag{3}$$

In the first row of $\Upsilon_k(\pi)$ we calculate the completion times of all the jobs in $\pi$. In the second row we ensure that the given job $k \in J$ is critical in $\pi$ (i.e., it has the greatest weighted lateness in $\pi$). Finally, in the last row we check condition (2) for job $k$ (see Theorem 1). It is a direct consequence of Theorem 1 that the sequence $\pi$ is optimal if and only if at least one of the systems $\Upsilon_k(\pi)$, $k \in J$, is uncontradictory (i.e., all conditions in $\Upsilon_k(\pi)$ are fulfilled). Now, we use this fact to show how to carry out the sensitivity analysis in the considered problem. Let $P = \{\alpha_1, \ldots, \alpha_l\} \subseteq \Phi$ be a given subset of parameters. Let us define the set $V_k(P; \pi)$, $k \in J$, which consists of all the vectors of parameters $(\alpha_1, \ldots, \alpha_l) \in \mathbb{R}^l$ for which the system $\Upsilon_k(\pi)$ is uncontradictory. We obtain the characterization of the set $V_k(P; \pi)$ by simply fixing the values of the parameters from the set $\Phi \backslash P$ in (3) and by simplifying the resulting system. If $P$ consists of only one parameter, then $V_k(P; \pi)$ is an interval in $\mathbb{R}$. If $|P| > 1$ then $V_k(P; \pi)$ is described by some simple relations between the parameters in the set $P$. Let us define:

$$V(P; \pi) = \bigcup_{k \in J} V_k(P; \pi) . \tag{4}$$

The set $V(P; \pi)$ consists of all the vectors of parameters $(\alpha_1, \ldots, \alpha_l)$ for which the sequence $\pi$ is optimal, under the assumption that all the other parameters (i.e., parameters from the set $\Phi \backslash P$) are fixed. In order to prove this fact assume that $(\alpha_1, \ldots, \alpha_l) \in V(P; \pi)$. From (4) we obtain that there exists $k \in J$ such that $(\alpha_1, \ldots, \alpha_l) \in V_k(P; \pi)$, which means that the system $\Upsilon_k(\pi)$ is uncontradictory for $(\alpha_1, \ldots, \alpha_l)$. From the sufficient condition in Theorem 1 we obtain that $\pi$ is optimal in this case. On the other hand, assume that $\pi$ is optimal for $(\alpha_1, \ldots, \alpha_l) \in \mathbf{R}^l$. From the necessary condition in Theorem 1 we obtain that there must exist $k \in J$ for which the system $\Upsilon_k(\pi)$ is uncontradictory for $(\alpha_1, \ldots, \alpha_l)$. This means that $(\alpha_1, \ldots, \alpha_l) \in V_k(P; \pi)$ and by (4) it holds $(\alpha_1, \ldots, \alpha_l) \in V(P; \pi)$. If the set $P$ consists of only one parameter, then $V(P; \pi)$ is a sum of at most $n$ intervals and it can be easily calculated in polynomial time. If $|P| > 1$ then the description of $V(P; \pi)$ may be more complicated.

**Example 1.** Let $J = \{1, \ldots, 6\}$, $prec = \{3 \to 4, 2 \to 6\}$, with all the parameters $p_i$, $d_i$, $w_i$, $i \in J$, being given in Table 1.

We carry out the sensitivity analysis for $\pi = (1, 2, 3, 4, 5, 6)$. First, we perform the sensitivity analysis for the single parameter $p_3$, i.e., we are going to calculate the set $V(\{p_3\}; \pi)$. From (4) it follows that we have to calculate all the sets $V_k(\{p_3\}; \pi)$, $k \in J$. As an example we show how to obtain $V_5(\{p_3\}; \pi)$. From the definition of $V_k(P; \pi)$ it follows that the set $V_5(\{p_3\}; \pi)$ consists of all the values of $p_3$, which fulfill the system $\Upsilon_5(\pi)$, assuming that all the other parameters are fixed (their values are taken from

Table 1). So, we fix all the parameters apart from $p_3$ in $\Upsilon_5(\pi)$ (see (3)). We get $C_1 = 8$, $C_2 = 12$, $C_3 = 12 + p_3$, $C_4 = 14 + p_3$, $C_5 = 19 + p_3$, $C_6 = 21 + p_3$. We also have $\mathbf{S}_5(\pi) = \{1, 2, 4\}$. From (3) we obtain:

$$\Upsilon_5(\pi) = \begin{cases} 0.3(19 + p_3 - 6) \geq 1.1(8 - 5), \\ 0.3(19 + p_3 - 6) \geq 1(12 - 10), \\ 0.3(19 + p_3 - 6) \geq 1.4(12 + p_3 - 13), \\ 0.3(19 + p_3 - 6) \geq 0.7(14 + p_3 - 15), \\ 0.3(19 + p_3 - 6) \geq 0.2(21 + p_3 - 12), \\ 0.3(19 + p_3 - 6) \leq 1.1(19 + p_3 - 5), \\ 0.3(19 + p_3 - 6) \leq 1(19 + p_3 - 10), \\ 0.3(19 + p_3 - 6) \leq 0.7(19 + p_3 - 15). \end{cases} \tag{5}$$

**Table 1**

Parameters of the sample problem

| $i \in J$ | $p_i$ | $d_i$ | $w_i$ |
|---|---|---|---|
| 1 | 8 | 5 | 1.1 |
| 2 | 4 | 10 | 1 |
| 3 | 3 | 13 | 1.4 |
| 4 | 2 | 15 | 0.7 |
| 5 | 5 | 6 | 0.3 |
| 6 | 2 | 12 | 0.2 |

Solving (5) we obtain $p_3 \in [2.75, 4.82]$, so $V_5(\{p_3\}; \pi) = [2.75, 4.82]$. In the same way we obtain that $V_3(\{p_3\}; \pi) = [4.82, 8.5]$ and $V_k(\{p_3\}; \pi) = \varnothing$ for $k \in \{1, 2, 4, 6\}$. Finally, $V(\{p_3\}; \pi) = V_3(\{p_3\}; \pi) \cup V_5(\{p_3\}; \pi) = [2.75, 8.5]$. It means that $\pi$ remains optimal if and only if $p_3$ belongs to the interval [2.75, 8.5].

In the same way it is possible to perform the sensitivity analysis for more than one parameter, for example, for $p_3$ and $w_3$. We obtain:

$$\begin{aligned} &V_1(\{ p_3, w_3\}; \pi) = \varnothing, \\ &V_2(\{ p_3, w_3\}; \pi) = \varnothing, \\ &V_3(\{ p_3, w_3\}; \pi) = \{( p_3, w_3): p_3 \geq 0, w_3 \geq 0, 4.8 \leq w_3 ( p_3 - 1) \leq 5\}, \\ &V_4(\{ p_3, w_3\}; \pi) = \varnothing, \\ &V_5(\{ p_3, w_3\}; \pi) = \{( p_3, w_3): p_3 \geq 2.75, w_3 \geq 0, w_3 ( p_3 - 1) \leq 3.9 + 0.3p_3\}, \\ &V_6(\{ p_3, w_3\}; \pi) = \varnothing. \end{aligned} \tag{6}$$

From (4) and (6) we get

$$V(\{\ p_3, w_3\};\ \pi) = V_3(\{\ p_3, w_3\};\ \pi) \cup V_5(\{\ p_3, w_3\};\ \pi).$$

In this case, the set $V(\{\ p_3, w_3\};\ \pi) \subset \mathbb{R}^2$ is described by some relations between parameters $p_3$ and $w_3$.

## 4. The problem $1|outtree|\Sigma w_iC_i$

The set of parameters in this problem includes: nonnegative processing times $p_i$ and positives weights $w_i$ given for each job $i \in J$, so $\Phi = \{p_1, \ldots, p_n, w_1, \ldots, w_n\}$. We assume that the precedence constraints in the problem are of the *outtree* type. This means that each job in the graph of the precedence constraints has at most one direct predecessor (see, for example, Fig. 1). This assumption is necessary since the more general problem, in which the precedence constraints are unrestricted, is strongly *NP*-hard even if $w_i = 1$, $i \in J$ (see [2]). Let us denote by $C_i(\pi)$ the completion time of the $i$-th job in a given sequence $\pi$. The cost function takes the following form:

$$F(\pi) = \sum_{i \in J} w_i C_i(\pi),$$

thus the objective is to minimize the weighted sum of completion times. The problem can be solved in $O(n^2)$ time (see [1]). Let us now present some additional results on the problem considered. These results are presented in detail in [3]. Consider a feasible sequence $\pi$. Any subsequence of jobs in $\pi$ is called a *block*. For example, if $\pi = (2, 3, 1, 4, 5)$ then (2, 3, 1), (1, 4) and (5) are blocks in $\pi$. We say that a block $\sigma = (\sigma(1), \ldots, \sigma(u))$ is *simple* if it consists of only one job or $\sigma(1)$ must precede all the jobs $\sigma(2), \ldots, \sigma(u)$ (in other words, if there is a path from job $\sigma(1)$ to each job $\sigma(2), \ldots, \sigma(u)$ in the graph of the precedence constraints). Assume that $\sigma_1$and $\sigma_2$ are two different blocks in a feasible sequence $\pi$. We say that $\sigma_1 \sim \sigma_2$ if and only if the following conditions hold:

1. Block $\sigma_2$ is processed just after block $\sigma_1$ in $\pi$, so $\pi = (\rho_1, \sigma_1, \sigma_2, \rho_2)$, where $\rho_1$ and $\rho_2$ are blocks or empty sequences.
2. Blocks $\sigma_1$ and $\sigma_2$ can be exchanged without violating the precedence constraints, i.e., the sequence $\pi' = (\rho_1, \sigma_2, \sigma_1, \rho_2)$ is feasible.

The following theorem holds [3]:

**Theorem 2.** *Assume that $\pi$ is a feasible sequence. Then $\pi$ is optimal if and only if for all simple blocks $\sigma_1$ and $\sigma_2$ in $\pi$ such that $\sigma_1 \sim \sigma_2$ holds:*

$$\frac{\Sigma_{i\in\sigma_1} p_i}{\Sigma_{i\in\sigma_1} w_i} \le \frac{\Sigma_{i\in\sigma_2} p_i}{\Sigma_{i\in\sigma_2} w_i}. \tag{7}$$

Let **SB**($\pi$) denote the set of all pairs of simple blocks ($\sigma_1$, $\sigma_2$) in $\pi$ such that $\sigma_1 \sim \sigma_2$. The set **SB**($\pi$) contains at most $O(n^2)$ elements and in [3] a polynomial algorithm for calculating the set **SB**($\pi$) is presented. Having the set **SB**($\pi$) we can construct the system of |**SB**($\pi$)| inequalities of the form (7). By Theorem 2 the sequence $\pi$ is optimal if and only if all the inequalities in the resulting system are fulfilled. Taking advantage of this fact we show how to perform the sensitivity analysis for a given sequence $\pi$. Assume that $P = \{\alpha_1, \ldots, \alpha_l\} \subseteq \Phi$ is a given subset of parameters. We obtain the characterization of the set $V(P; \pi)$ by fixing the values of all the parameters from $\Phi \backslash P$ in (7). If $|P| = 1$ then $V(P; \pi)$ is an interval in $\mathbb{R}$. If $|P| > 1$ then $V(P; \pi)$ is described by some simple relations between parameters in $P$.

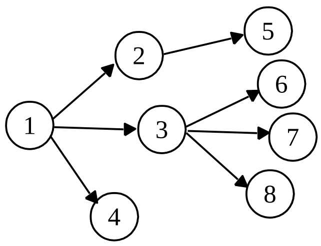


**Fig. 1.** Graph of the precedence constraints in the sample problem

**Table 2**

Parameters of the sample problem

| $i \in J$ | $p_i$ | $w_i$ |
|---|---|---|
| 1 | 9 | 2 |
| 2 | 18 | 3 |
| 3 | 8 | 3 |
| 4 | 7 | 4 |
| 5 | 14 | 1 |
| 6 | 12 | 2 |
| 7 | 18 | 3 |
| 8 | 9 | 4 |

**Example 2.** Consider a problem in which $J = \{1, 2, \ldots, 8\}$, with all the parameters $p_i$ and $w_i$, $i \in J$ being given in Table 2. The graph of the precedence constraints is given in Figure 1. We carry out the sensitivity analysis for the optimal sequence $\pi$ = (1, 4, 3, 8, 7, 6, 2, 5).

First we calculate the set $\mathbf{SB}(\pi)$:

$$\mathbf{SB}(\pi) = \{((4), (3)), ((4), (3,8)), ((4), (3,8,7)), ((4), (3,8,7,6)), ((3,8,7,6), (2)), ((3,8,7,6), (2,5)), ((8), (7)), ((7), (6)), ((6), (2))\ ((6), (2,5))\}\ .$$

Let us show, for example, why $((3,8,7,6), (2,5)) \in \mathbf{SB}(\pi)$. It is easy to see that (3,8,7,6) and (2,5) are two neighbour blocks in $\pi$, which can be exchanged without violating the precedence constraints (since jobs 2 and 5 have no predecessors in the set {3,8,7,6}). Block (2,5) is simple since 2 must precede job 5 and block (3,8,7,6) is simple since job 3 must precede jobs 8, 7 and 6 (see Fig. 1).

Having the set $\mathbf{SB}(\pi)$ we construct inequalities (7) for the problem under consideration:

$$\begin{aligned}
&w_3 p_4 \le w_4 p_3 \\
&(w_3 + w_8)p_4 \le w_4( p_3 + p_8) \\
&(w_3 + w_8 + w_7)p_4 \le w_4( p_3 + p_8 + p_7) \\
&(w_3 + w_8 + w_7 + w_6)p_4 \le w_4( p_3 + p_8 + p_7 + p_6) \\
&w_2( p_3 + p_8 + p_7 + p_6) \le (w_3 + w_8 + w_7 + w_6)\, p_2 \\
&(w_2 + w_5)( p_3 + p_8 + p_7 + p_6) \le (w_3 + w_8 + w_7 + w_6)( p_2 + p_5) \\
&w_7 p_8 \le w_8 p_7 \\
&w_6 p_7 \le w_7 p_6 \\
&w_2 p_6 \le w_6 p_2 \\
&(w_2 + w_5)p_6 \le w_6( p_2 + p_5)
\end{aligned} \tag{8}$$

It follows from Theorem 2 that $\pi$ is optimal if and only if the parameters of the problem fulfil the system (8). Assume that we want to perform the sensitivity analysis for $w_3$ and $p_3$. We start with fixing all the parameters apart from $p_3$ and $w_3$ in (8). We obtain:

$$\begin{aligned}
&7w_3 \le 4p_3 \\
&7(w_3 + 4) \le 4( p_3 + 9) \\
&7(w_3 + 7) \le 4( p_3 + 27) \\
&7(w_3 + 9) \le 4( p_3 + 39) \\
&3( p_3 + 39) \le 18(w_3 + 9) \\
&8( p_3 + 39) \le 32(w_3 + 9)
\end{aligned} \tag{9}$$

Simplifying the system (9) we obtain:

$$V(\{p_3, w_3\}, \pi) = \left\{ (p_3, w_3) : \frac{1}{4} p_3 + \frac{3}{4} \le w_3 \le \frac{4}{7} p_3,\ p_3 \ge 0 \right\}. \tag{10}$$

Setting $p_3 = 8$ in (10) we obtain $V(\{w_3\}; \pi) = \left[ 2\frac{3}{4}, 4\frac{4}{7} \right]$. Similarly, setting $w_3 = 3$ in (10) we get $V(\{p_3\}; \pi) = \left[ 5\frac{1}{4}, 9 \right]$. This means that the sensitivity range for $w_3$ equals $\left[ 2\frac{3}{4}, 4\frac{4}{7} \right]$ and the sensitivity range for $p_3$ equals $\left[ 5\frac{1}{4}, 9 \right]$.

## 5. The problem $F2\,||C_{\max}$

In this section, we consider the two machine flow shop sequencing problem denoted by $F2||C_{\max}$ in Graham's notation. The set of jobs $J = \{1, \ldots, n\}$ must be processed on two machines $M_1$ and $M_2$. Each job $i \in J$ is processes first on the machine $M_1$ and then on the machine $M_2$. The order of processing is the same on both machines and it can be represented by a sequence of jobs $\pi$. For each job $i \in J$ there are given nonnegative processing times $a_i$ and $b_i$, respectively, on machines $M_1$ and $M_2$, thus $\Phi = \{a_1, \ldots, a_n, b_1, \ldots, b_n\}$. It is assumed that there are no precedence constraints between jobs, so each sequence of jobs is feasible. The objective is to calculate a sequence $\pi$ for which the completion time of the last job is minimal. The problem considered can be solved in $O(n \log n)$ time by famous Johnson's algorithm [7]. For each sequence $\pi$ it is possible to construct the activity network presented in Figure 2. It is easy to observe that the completion time $T(\pi)$ of the last job in $\pi$ (i.e., the job $\pi(n)$) is equal to the length of the longest path from the vertex $a_{\pi(1)}$ to the vertex $b_{\pi(n)}$. The number of paths from $a_{\pi(1)}$ to $b_{\pi(n)}$ equals $n$. Thus, $T(\pi)$ can be calculated as follows:

$$T(\pi) = \max_{k \in J} \left( \sum_{i=1}^{k} a_{\pi(i)} + \sum_{i=k}^{n} b_{\pi(i)} \right). \tag{11}$$

$$\begin{tikzcd}
a_{\pi(1)} \arrow[r] \arrow[d] & a_{\pi(2)} \arrow[r] \arrow[d] & \cdots \arrow[r] & a_{\pi(n)} \arrow[d] \\
b_{\pi(1)} \arrow[r] & b_{\pi(2)} \arrow[r] & \cdots \arrow[r] & b_{\pi(n)}
\end{tikzcd}$$

**Fig. 2.** The activity network for sequence $\pi$

A job $\pi(r) \in J$, $r = 1, \ldots, n$, is called *critical* in a given sequence $\pi$ if the following condition holds:

$$T(\pi) = \sum_{i=1}^{r} a_{\pi(i)} + \sum_{i=r}^{n} b_{\pi(i)} . \tag{12}$$

Thus, job $\pi(r)$ is critical if and only if $a_{\pi(1)} \to \ldots \to a_{\pi(r)} \to b_{\pi(r)} \to \ldots \to b_{\pi(n)}$ is the longest path in the activity network for $\pi$. Note that there may appear more than one critical job in $\pi$. The following theorem holds [6]:

**Theorem 3.** *Sequence $\pi$ is optimal if and only if there exists in $\pi$ a critical job $\pi(r)$ such that for all $i \in \{1, \ldots, r\}$ and for all $j \in \{r, \ldots, n\}$ holds:*

$$\min\{a_{\pi(i)}, b_{\pi(j)}\} \le \min\{a_{\pi(j)}, b_{\pi(i)}\} . \tag{13}$$

Now, we show how to apply Theorem 3 to the sensitivity analysis. Let $\pi$ be a given sequence of jobs let $r \in \{1, \ldots, n\}$. Consider the following system of inequalities $\Omega_r(\pi)$:

$$\Omega_r(\pi) = \begin{cases} \Sigma_{i=1}^{r} a_{\pi(i)} + \Sigma_{i=r}^{n} b_{\pi(i)} \ge \Sigma_{i=1}^{j} a_{\pi(i)} + \Sigma_{i=j}^{n} b_{\pi(i)} & \text{for } j = 1, \ldots, n, \\ \min\{a_{\pi(i)}, b_{\pi(j)}\} \le \min\{a_{\pi(j)}, b_{\pi(i)}\} & \text{for } i \in \{1, \ldots, r\},\ j \in \{r, \ldots, n\}. \end{cases} \tag{14}$$

The system $\Omega_r(\pi)$ consists of $n + r(n - r + 1)$ inequalities. In the first row of $\Omega_r(\pi)$ we check whether the job occupying the $r$th position in $\pi$ is critical. In the second row of $\Omega_r(\pi)$ we check conditions (13) from Theorem 3. It is clear that the sequence $\pi$ is optimal if and only if at least one of the systems $\Omega_r(\pi)$, $r = 1, \ldots, n$, is uncontradictory. Now, we can perform the similar analysis as in Section 2. For a given subset of parameters $P = \{\alpha_1, \ldots, \alpha_l\} \subseteq \Phi$ we define the set $V_r(P; \pi)$, $r = 1, \ldots, n$, which consists of all the vectors of parameters $(\alpha_1, \ldots, \alpha_l)$ for which the system $\Omega_r(\pi)$ is uncontradictory (assuming that all the parameters from the set $\Phi \backslash P$ are fixed). Finally, the set $V(P; \pi)$ can be calculated as follows:

$$V(P; \pi) = \bigcup_{r=1}^{n} V_r(P; \pi) . \tag{15}$$

Using the same argumentation as in Section 3 we can prove that the set $V(P; \pi)$ consists of all the vectors of parameters $(\alpha_1, \ldots, \alpha_l)$ for which the sequence $\pi$ is optimal.

**Example 3.** Consider a problem in which $J = \{1, 2, \ldots, 6\}$ and all the parameters $a_i$ and $b_i$, $i \in J$ are given in Table 3.

**Table 3**

Parameters of the sample problem

| $i \in J$ | $a_i$ | $b_i$ |
|---|---|---|
| 1 | 3 | 6 |
| 2 | 7 | 4 |
| 3 | 4 | 7 |
| 4 | 5 | 3 |
| 5 | 7 | 3 |
| 6 | 2 | 2 |

We perform the sensitivity analysis for the optimal sequence $\pi = (1, 2, 3, 4, 5, 6)$ and for the parameters $a_3$ and $b_3$, so we want to characterize the set $V(\{a_3, b_3\}; \pi)$. First, we calculate all the sets $V_r(\{a_3, b_3\}; \pi)$, $r = 1, \ldots, n$ (see (15)) by fixing all parameters apart from $a_3$ an $b_3$ in $\Omega_r(\pi)$. After simplification we get $V_r(\{a_3, b_3\}; \pi) = \varnothing$ for $r \in \{1, 2, 3, 4, 6\}$ and

$$\begin{aligned} V_5(\{a_3, b_3\}; \pi) = \{(a_3, b_3) : b_3 \in [0, 9] a_3 \geq 0, b_3 - 5 \leq a_3 \leq b_3\} \cup \\ \{(a_3, b_3) : b_3 \in [3, 9] a_3 \geq 0, b_3 - 5 \leq a_3\}. \end{aligned} \tag{16}$$

From (15) it follows that $V(\{a_3, b_3\}; \pi) = V_5(\{a_3, b_3\}; \pi)$ and $V(\{a_3, b_3\}; \pi)$ consists of all the vectors $(a_3, b_3)$ for which $\pi$ is optimal. Setting $a_3 = 4$ in (16) we obtain $V(\{b_3\}; \pi) = [3, 9]$ and setting $b_3 = 7$ in (16) we get $V(\{a_3\}; \pi) = [2, \infty)$.

## 6. Conclusions

In this paper, it is shown how to perform the sensitivity analysis for three particular sequencing problems. For all of the considered problems it is possible to give a full characterization of the set $V(P; \pi)$. In particular, it is easy to calculate the sensitivity range for each, single parameter. Sensitivity analysis is very important from the practical point of view. The assumption that all the parameters are known in advance is not valid for most of the real-world processes and it restricts the practical aspect of sequencing. Sensitivity analysis is one of the several possibilities of taking the precision into account. The obtained results may also be used as a background to investigate stochastic or fuzzy sequencing problems (see [5]). The work in the field of sensitivity analysis should be continued. The sensitivity analysis requires a particular problem to by analyzed more deeply. A good starting points is to formulate some sufficient and necessary conditions of optimality of a given sequence. This is the case for all the sequencing problems studied in this paper (see Theorems 1, 2 and 3).

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## Analiza wrażliwości w problemach kolejnościowych

Problem kolejnościowy jest specjalnym przypadkiem ogólniejszego problemu szeregowania. W problemie kolejnościowym celem podejmującego decyzję jest wyznaczenie dopuszczalnej kolejności (permutacji) prac, dla której wartość zadanej funkcji kosztu jest najmniejsza. W typowym problemie kolejnościowym zadane są pewne parametry (np. czas trwania prac), których wartości muszą być ustalone przed wyznaczeniem optymalnego rozwiązania. Po wyznaczeniu optymalnej kolejności prac istotne może być pytanie o stabilność otrzymanego rozwiązania. Można zapytać, w jakim zakresie mogą się zmieniać wartości parametrów problemu, aby otrzymane rozwiązanie pozostało optymalne? Taka analiza nazywana jest analizą wrażliwości. Aby otrzymać efektywną metodę przeprowadzenia analizy wrażliwości, każdy szczególny przypadek problemu kolejnościowego należy badać osobno. W artykule przedstawiono efektywne metody przeprowadzenia analizy wrażliwości dla trzech wybranych problemów kolejnościowych.